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# A Novel Approach to Reproduced Kernel Hilbert Space for Artificial Intelligence Control: Using Monte-Carlo Estimates of Operators Arising

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**Abstract—**An embedding of stochastic optimal control problems of artificial intelligence form into reproducing kernel Hilbert spaces is presented in this study. A model-free, non-parametric approach for calculation of an approximate solution to the control problem is obtained by consistent, sample based estimates of the embedding. The solve the problem; it decomposes into two components; invariant and task dependent. Hence, the solution presented in the current paper used sample data more efficiently than previous sample based approaches in this field by some innovations such as allowing sample re-use across tasks. To show the efficiency of the introduced approach, numerical examples on test problems are presented.

**Index Terms**— Monte-Carlo Estimates, Operators, Artificial Intelligence, Hilbert Space, Reproducing Kernel, Double Slit, Applied Mathematics



## 1 Introduction

In spite of challenges in solving general non-linear stochastic optimal control (SOC) and Reinforcement Learning (RL) problems, a type of problems which can be solved by closed form solutions are recently identified [1-8]. To solve these problems, it is necessary to evaluate an artificial intelligence, which is equivalent to evaluating a partition function which in turn is a tough problem. However, these solutions are interesting due to their several practical applications, resulting from the possibility of applying Monte-Carlo and variational methods for solving these problems [9, 10]. Analytical evaluating the required artificial intelligence, based on linear operators acting on state vectors, is possible in some special cases such as linear dynamics and quadratic costs [11, 14, 16, and 17]. It is shown in the current paper that the artificial intelligence can be evaluated, regarding the covariance operators acting on elements of the Hilbert space, by appropriate embedding of it into a reproducing kernel Hilbert space (RKHS). Although a tractable solution to the SOC problem does not yield in itself, consistent estimators of the required operators lead to efficient non-parametric algorithms [11-23].

A critical prominence of estimating the operators other than directly estimating the artificial intelligence, as the goal of the previous applications of Monte-Carlo methods, is that some deficiencies of previous methods can be eliminated without considerable loss of their advantages [24]. In this approach, the complexity of sample is considerably reduced due to separating the problem into an invariant and task varying

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International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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component [25]. It allows efficient sample re-use across tasks and leads to a form of transfer learning which in turn, leads to the situation where any change in the task including, for e.g., different start states, necessitate acquiring new samples [26-39]. Moreover, as the approach remains model-free, it is applicable to the RL setting, unlike to variational [40-45] or function approximation [46-59] approaches. As a result, it is further distinguished through convergence guarantees. The operators becomes state-dimensionality independent by embedding of RKHS. It leads to better scalability of operators. However, by informing choices of sampling procedures and kernel, incorporating the prior knowledge about both tasks and dynamics is effectively possible [60].

It should be noted that the presented approach in this study is not limited to problems which are in context of artificial intelligence stochastic optimal control. For solving linearly solvable MDPs by [61, 62], inference control by [63-73] and free energy control of [74, 75], an underlying problem of equivalent form should be solved. As a result, the proposed methods are directly applicable in these fields. In addition, a generalization of artificial intelligence control to develop an optimal policy for general SOC problems is described by [76-83]. Moreover, [84] and [85] were proposed artificial intelligence formulations for discounted and average cost infinite horizon problems and risk sensitive controls, respectively, while in this study, finite horizon problems are critically focused.

A brief review of the formulation of a SOC problem in the artificial intelligence control framework of [86-90], which has two specific properties related to the presented approach in this study. The first of these characteristics is that this formulation solves the optimal policy according to a state desirability function $\Psi$ in a closed form manner. The second one is the possibility of expressing this function $\Psi$ as a conditional expectation of the product of an immediate cost and a future desirability. Therefore, the desirability function can be computed by identifying this expression with an inner product in a RKHS. The concept is formulated, as the model-based case is considered and the analytical form of the RKHS based evaluation also is extracted. Discussing about the possible estimating of this operator from transition samples, which leads to performing in form of a finite dimensional inner product, considering the model-free case. For solutions of the SOC problem are summarized as (i) transition samples gathered and then, embedding operator estimated, (ii) the desirability function and immediate cost represented as elements of RKHSs, (iii) the desirability function evaluated, recursively, as a series of inner products based on the estimated operator –including a series of matrix multiplications – and (iv) the optimal controls computed based on the obtained desirability function. The methodology is developed, accompanying by a series of alternative estimators so that the goal is either reducing the computational costs or allowing more efficient use of the sample data. The verifications of the proposed methods by experimental data.

## 2 General Description of Artificial Intelligence Control

The artificial intelligence approach to stochastic optimal control, proposed by [79-84] (see also [85-90]), is reviewed in the current section. Considering $x \in R^{d_x}$ as the system state and $u \in R^{d_u}$ as the control signals, a continuous time stochastic system can be taken into account as the following form:

$$dx = f\left(x,t\right)dt + B\left(x,t\right)\left(udt + d\zeta\right) \tag{1}$$

where $d\zeta$ is a multivariate Wiener process with $E\left[d^{\zeta^2}\right] = Q\left(x,t\right)dt$, and $f$, $B$ and $Q$ may be non-linear functions. It is worthwhile to note that the system is linear in the controls while both noise and controls act in the same subspace. By an objective of the following form, the best Markov policy, i.e., $u\left(t\right) = \pi\left(x\left(t\right),t\right)$ is searched:

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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$$J^{\pi}(x,t)=E_{X^{\pi}(0)|x}\left[C_{\bullet}\left(X^{\pi}(T)\right)+\int_{t}^{T}C\left(X^{\pi}(s),s\right)+u(s)^{T}\,Hu(s)ds\right] \quad (2)$$

where $T$ is a specified terminal time, $C_{\bullet}$ and $C$ are a terminal and running cost, respectively. The expectation is taken w.r.t. $X^{\pi}(0)$, the paths of Eq. (1) starting in $x$ and following policy $\pi$. By requiring the quadratic control cost, as $H\in R^{d_u\times d_u}$, to satisfy $Q=\lambda BH^{-1}B^{T}$ for some constant scalar $\lambda>0$, it is further constrained.

It was shown by [79-90] that the optimized objective for this form of problem can be stated as:

$$J^{*}(x,t)=\min_{\pi}J^{\pi}(x,t)=-\lambda\log\Psi(x,t) \quad (3)$$

where $\Psi$ is a state desirability function and can be obtained by the following path integral:

$$\Psi(x,t)=E_{X^{0}(0)|x}\left[e^{-\int_{t}^{T}\frac{1}{\lambda}C\left(X^{0}(s),s\right)ds}\Psi\left(X^{0}(T),T\right)\right] \quad (4)$$

with $\Psi(0,\mathrm{T})=\exp\left\{-C_{\bullet}(0)/\lambda\right\}$. The expectation in Eq. (4) is taken w.r.t. uncontrolled path of the dynamics Eq. (1), i.e. those under the policy $\pi^{0}(0,0)=0$, starting in $x_{t}$.

It can be described the optimal policy $\pi^{*}(x,t)$, directly, in terms of as the following due to linear control with quadratic control cost and $\Psi$:

$$\pi^{*}(x,t)=-H^{-1}B(x)^{T}\nabla_{x}J^{*}(x,t) \quad (5)$$

$$\pi^{*}(x,t)=H^{-1}B(x)^{T}\frac{\lambda\nabla_{x}\Psi(x,t)}{\Psi(x,t)} \quad (6)$$

As a result, obtaining $\Psi$ is the most challenging computation in this form of problem.

If the optimal controls at certain time points, say $\left\{t_{1,\ldots,\mathrm{n}}\right\}$ with $t_{n}=T$, is emphasized, a representation in terms of the finite dimensional distribution $\Psi_{i}(x)=\Psi(x,t_{i})$ can be obtained from Eq. (4), only by computing the set $X=\left(X^{0}(t_{0}),\ldots,\mathrm{X}^{0}(t_{n})\right)$. The following recursive expression can be specifically obtained using the Markov property of $X^{0}(t)$ and marginalising intermediate states:

$$\Psi_{i}(x_{t_i})=E_{X_{i+1}|x_{t_i}}\left[\Phi_{i}(x_{t_i},X_{i+1})\Psi_{i+1}(X_{i+1})\right] \quad (7)$$

where,

$$\Phi_{i}(x_{t_i},x_{t_{i+1}})=E_{X^{0}(0)|x_{t_i},x_{t_{i+1}}}\left[e^{-\frac{1}{\lambda}\int_{t_i}^{t_{i+1}}C\left(X^{0}(s),s\right)ds}\right] \quad (8)$$

with the expectation taken w.r.t. uncontrolled paths from $x_{t_i}$ to $x_{t_{i+1}}$. It should be pointed out that $-\lambda\log\Phi_{i}$ can be considered as the (optimal) expected cost for the problem ranging from $x_{t_i}$ to $x_{t_{i+1}}$ over the time horizon $[t_{i},t_{i+1}]$ under dynamics and running costs corresponding to those of the overall problem given in Eq. (2). Therefore, the problem naturally separated into simpler compounds; a set of short horizon problems – or indeed a nested hierarchy of such $\Phi-$ while, in the same time, it is a set of recursive evaluations backwards in time.

## 3 The Artificial Intelligence Embedding

Here, a demonstration is provided about the possibility of expressing Eq. (7) in terms of linear operators in RKHS. More details about the theory of RKHS and basic concepts of the presented theory in this paper can be found in [69-73] and [74-78] and [79-90], respectively.

Initially, the evaluation of a single step, i.e., $\Psi_{i}$ given $\Psi_{i+1}$, is considered and model-based analytical expressions for the evaluation of $\Psi_{i}$ in terms of certain operators in RKHS is extracted.

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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### 3.1 Embedding of the Artificial Intelligence by a Model-Based Analytical One Step Process

It shows that Eq. (7) may write as an inner product in a RKHS in the case based on the model. The process consists of three steps as expressing expectations in terms of inner products in a RKHS, firstly; adapting the basic expression to conditional expectations, secondly; and then, taking into account the conditional expectations of functions of both the conditional and conditioning variable.

$H^{\kappa}$ is generally used as the RKHS of functions $Z \to R$ associated with the positive semi-definite kernel $\kappa(0,0)$. Considering $P^{Z}$ as the set of random variables on $Z$, the embedding operator can be defined as $\varepsilon^{\kappa}: P^{z} \to H^{\kappa}$:

$$E_{Z}\left[h(Z)\right]=\left\langle h, \underbrace{E_{Z}\left[k(Z,0)\right]}_{:=\varepsilon^{\kappa}[Z]} \right\rangle$$

$$\forall Z \in P^{z}, h \in H^{\kappa} \tag{9}$$

in which the standard embedding of individual elements $z \in Z$ is directly extended into $H^{\kappa}$, given by $\varepsilon^{\kappa}[z]=\kappa(z,0)$ commonly encountered.

The main objective of the considered problem in this study is evaluating $\Psi_{i}$ given in Eq. (7); i.e. proper embedding of $X_{i+1}|x_{i}$ which makes expressing the required expectation as an inner product in some RKHS possible. As $X_{i+1}|x_{i}$ is a simple random variable for fixed $x_{i}$, direct applying of Eq. (9) is possible. However, considering a general conditional random variable $Z|y$, act as a map $Y \to P^{z}$ is more suitable. It yields random variables over $Z$ which gives a value $y \in Y$, and defines a conditional embedding $U^{l\kappa}: H^{l} \to H^{\kappa}$ s.t.:

$$\varepsilon^{\kappa}\left[Z|y\right]=U^{l\kappa} \circ \varepsilon^{l}\left[y\right] \tag{10}$$

It should be pointed out that $\varepsilon^{l}\left[y\right]=l(0,y)$. It is used in kernel methods as the standard embedding operator of elements $y \in Y$ used in kernel methods. Therefore, conditional expectations can be expressed as:

$$E_{Z|y}\left[h(Z)\right]=\left\langle h, \underbrace{E_{Z|y}\left[\kappa(Z,0)\right]}_{:=\varepsilon^{\kappa}[Z|y]=U^{l\kappa}[l(y,0)]} \right\rangle \tag{11}$$

were demonstrated the existence and a specific form of an operator $U$ which satisfies Eq. (10). It should be noted that the argument of the expectation in Eq. (7), and in particular of $\Phi$, is a function of both the random variable, i.e., $X_{i+1}$ and the conditioning $x_{i}$. It is opposite to $h$ in Eq. (11). Therefore, direct applying of Eq. (11) is not possible. However, an assisting random variable $\tilde{X}$ is introduced where $P\left(\tilde{X}, X_{i+1}|x_{i}\right)=P\left(X_{i+1}|x_{i}\right)\delta_{\tilde{X}=x_{i}}$ with $\delta$ the delta distribution. Therefore, the following can be written for all $h \in H^{\kappa}$:

$$E_{X_{i+1}|x_{i}}\left[h(x_{i}, X_{i+1})\right]=E_{X_{i+1},\tilde{X}|x_{i}}\left[h(\tilde{X}, X_{i+1})\right]$$

$$=\left\langle h, \varepsilon^{\kappa}\left[X_{i+1}, \tilde{X}|x_{i}\right]\right\rangle \tag{12}$$

If $x_{i}$ is considered as a constant parameter, an alternative formulation can be obtained with equivalent analytical setting, but without an immediate yielding of a practical empirical estimator.

In order to substitute the specific argument encountered in Eq. (7) for the generic function $h$, it is assumed that $H^{\psi}$, $H^{\phi}$, s.t. $\Psi \in H^{\psi}$, $\Phi \in H^{\phi}$, are given ($R^{d_x} \times R^{d_x} \to R$ and $R^{d_x} \to R$ are a space of functions $H^{\phi}$ and

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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functions $H^{\psi}$, respectively). The mismatch in the arity of functions in these spaces can be considered by extending $H^{\psi}$ to $H^{\psi'}$; a space of functions $R^{d_x} \times R^{d_x} \to R$, using the kernel $\psi'\left((u,v),(u',v')\right)=\psi(u,u')$. It means that $H^{\psi}$ and its tensor product with the RKHS of constant functions are identified. Then, Eq. (7) can be rewritten by taking the embedding of $X_{i+1}, \tilde{X} \mid x_i$ into $H^{\omega}=H^{\phi}\otimes H^{\psi'}$ in which the product function of $\Phi_i$, $\Psi_{i+1}$ locates, and using Eqs. (11) and (12):

$$\Psi_i(x)=E_{X_{i+1}|X_i}=x\left[\Phi_i(X_{i+1},x).\Psi_{i+1}(X_{i+1})\right]$$

$$=\left\langle \Phi_i\otimes\Psi_{i+1},\varepsilon^{\omega}\left[X_{i+1},\tilde{X}\mid X_i=x\right]\right\rangle$$

$$=\left\langle \Phi_i\otimes\Psi_{i+1},\mathrm{U}^{\omega\kappa}\circ\varepsilon^{\kappa}[x]\right\rangle \quad (13)$$

where $k$ is some kernel over chosen $R^{d_x}$. Taking $\kappa$ so that it be able to reuse the pre-computed matrices during recursive evaluation of $\psi$ estimates of $\Psi$ is computationally critical (see Eq. (16)).

### 3.2 Estimations of Model-Free Finite Sample

To evaluate the embedding of random variables $U$ and artificial intelligence Eq. (13), it is necessary that expectations of kernels to be evaluated and remained inflexible. At the same time, there should be a detailed analytical knowledge about the system dynamics Eq. (1). However, forming empirical estimates is simple and it leads to practical algorithms.

Considering $D=\left\{(x,x')_{1\ldots m}\right\}$ as a set of i.i.d, transition samples of the uncontrolled dynamics, e.g., a sample set obtained from trajectory executions under the policy $\pi^0$, can be expressed. [80-90] were shown that a regularized estimate of $U^{\kappa\omega}$ can be defined as:

$$\hat{U}^{\psi\omega}=g_D^{\omega}\left(G_{\chi\chi}^{\psi}+\varepsilon mI\right)^{-1}g_{\chi}^{\psi} \quad (14)$$

where $\varepsilon$ is a regularization parameter and $g_{\chi}^{\psi}, g_{\chi'}^{\omega}$ and $G_{\chi\chi}^{\psi}$ are the vectors of embeddings and Gramian on the sample data $D$ respectively, i.e., $\left[g_{\chi}^{\psi}\right]_i=\phi(x_i,0)$ and $\left[G_{\chi\chi}^{\psi}\right]_{ij}=\psi(x_i,x_j)$. The representations of $\Phi_i$ and $\Psi_{i+1}$ in their respective RKHSs are necessary for evaluating Eq. (13). It can be assumed, due to recursive evaluation $\Psi$, that the empirical estimate of $\Psi_{i+1}$ is defined as $\bar{\Psi}_{i+1}=\sum_{x_j\in\chi}[\alpha_i+1]_j\,\psi(x_j,0)=g_{\chi}^{\psi}\alpha_i$ where $\alpha_{i+1}$ is a vector of weights. In the same way, the representation of $\Phi_i$ in $H^{\phi}$ will be assumed as $g_B^{\phi}\beta$) for some weights $\beta$ and $B$ will be set. It will be shown that despite assurance from existence of such a representation with assuming that $\Phi_i\in H^{\phi}$, explicit computing of it is not necessary. By substituting the empirical operator $\hat{U}^{\kappa\omega}$ of Eq. (14) and the kernel expansions of $\Phi_i$ and $\Psi_{i+1}$ into Eq. (13), matrix algebra yields the empirical estimate of $\Psi_i(x)$ as:

$$\bar{\Psi}_i(x)=\left\langle\psi(x,0),g_{\chi}^{\psi}\alpha_i\right\rangle=G_{x\chi}^{\psi}\alpha_i \quad (15)$$

with weights $\alpha_i$ given by:

$$\alpha_i=\left[\underbrace{G_{DB}^{\phi}\beta}_{=\Phi_i(\chi,\chi')}\odot\underbrace{G_{\chi'A}^{\psi}\alpha_{i+1}}_{=\bar{\Psi}_{i+1}(\chi')}\right]^{T}\left(G_{\chi\chi}^{\psi}+\varepsilon mI\right)^{-1} \quad (16)$$

where $\odot$, denotes the Hadamard product. In addition to obtaining corresponding representation of $\Phi_i$ given by $\beta$), Eq. (16) can be used to recursive evaluation of the weights $\alpha$. It should be noted that the term involving the representation of $\Phi_i$ can be written as:

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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$$G_{DB}^{\phi}\beta = \Phi_i\left(\chi,\chi'\right) = \left(\Phi_i\left(x_1,x_1'\right),\Phi_i\left(x_1,x_2'\right),...\right)^T \quad (17)$$

This representation states that it is not necessary to obtain an explicit representation of $\Phi$ in terms of the kernel expansion in $H^{\phi}$ but the ability to evaluate $\Phi_i$ at the sample data $D$, which is comparable to evaluating the cost function, is sufficient.

It should be critically noted that direct recursive computation of all $\bar{\Psi}_{1...n}$ is possible since $\bar{\Psi}_i$ is a finite weighted sum of kernels and therefore, $\bar{\Psi}_i \in H^{\psi}$. In addition, pre-computing of all required matrices is possible since those are only functions of the sample data. Hence, it is easy to obtain an approximate optimal policy from Eq. (5) for fine discretisations of the problem.

## 4 Efficient Estimators

High computational complexity of $O\left(m^3\right)$ for the matrix inversion is one of main shortcomings of the basic estimator Eq. (16). If the same $D$ and $O\left(m^2\right)$ per iteration are used in each time step, it is required. Another deficiency of Eq. (16) is that it needs to sample data under the uncontrolled dynamics. Therefore, off policy learning is not allowed. Alternative estimators for $U$ by low rank approximations or importance sampling are used to remove these drawbacks. We choose to omit the discussion of these in order to address a, in our opinion, often overlooked aspect of efficiency when solving varying problems under the same dynamics. Actually, there are not individual tasks to be solved but several related problems should be solved in a repeated manner. For example, use of an optimized single reaching movement is limited due to need for a series of such movements with changing start and target states as a result of complicated interactions. The solution for this problem in previous methods is generally based on the re-initialization for each specified problem. For instance, the start state of Monte-Carlo method is a new sample set, even when trivial changes are performed. Some extensions to the proposed method which aims to improve the efficiency of sampling as samples can reuse over and over for repeated applications are discussed in the following section.

### 4.1 Using Transition Sample Re-Use for Transfer Learning

One of the practical problems of estimators is the necessity of evaluating $\Phi$ at the training transitions. Hence, the favorite is obtaining an estimator based on evaluation of $\Phi$ on a separate, ideally arbitrary, data set $D'$. It can be seen that:

$$G_{DB}^{\phi}\beta = \left\langle \Phi, \phi\left(D,0\right)\right\rangle = \left\langle \Phi, C_{ZZ}^{\phi\phi}\left(C_{ZZ}^{\phi\phi}\right)^{-1}\phi\left(D,0\right)\right\rangle$$

$$\approx \underbrace{\beta^T G_{B'D'}^{\phi}}_{\Phi(D')}\left(G_{D'D'}^{\phi} + \varepsilon m' I\right)^{-1} G_{D'D}^{\phi}$$

where $Z$ is some free random variable with support on $R^{d_x} \times R^{d_x}$, which is implemented as an empirical estimator based on a data set $D' = \left\{\left(x, x'\right)_{1...m'}\right\}$ of i.i.d. samples from $Z$ (often in practice $D' \subseteq D$). The considered result can be simply achieved by substituting into Eq. (16) since the indicated evaluation of the r.h.s. only needs evaluation of $D'$ at elements of $\Phi$. Specifically, the ability of pre-computing and reusing the inverse matrix of Eq. (16) across changing tasks is obtained in addition to across different time steps by an assumed time stationary dynamics. This is of importance for efficient estimation in, e.g., the RL setting where incurred costs are known only at observed transitions or in cases where $\Phi$ can be freely evaluated but it is expensive to do so. However, generating large sets of transition samples may be comparatively cheap, e.g., the case of simple kinematic control where

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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cost evaluation requires collision detection. Note that this form makes explicit use of the kernel $\phi$, and while we may not be able to guarantee $\Phi \in H^{\phi}$, by choosing a kernel such that the projection of $\Phi$ onto $H^{\phi}$ is close to $\Phi$, we can expect good results.

### 4.2 Sampling by Augmenting Task

Generally, samples should be collected from the task agnostic dynamics $X^{0}$. However, a task often induces regularities which suggest more suitable sampling distributions. To concentrate samples in regions of high $\Phi$, considering the role $\Phi$ takes in Eq. (16) (via Eq. (17)) as a weight vector appears to be advantageous similar to importance sampling. It is clear that $\Phi$ is a suitable guidance for choosing the sampling distribution. However, for repeated task, $\Phi$ can be incorporated, partly, into the sampling process which allows incremental learning of the task.

For executing several tasks of a generic skill, it is frequently characterized by two components as an invariant cost relating to the skill and a task specific cost components. Assuming that the state cost decomposes as:

$$C\left(x,\theta,t\right)=C_{skill}\left(x,t\right)+C_{task}\left(x,t,\theta\right) \tag{18}$$

where $\theta$ parametrises the task. In this case, the path integral Eq. (4) can be rewritten as:

$$\Psi = E_{X^{\nu}(0)|x_t}\left[ e^{-\int_t^T \frac{1}{\lambda} C_{task}\left(X^{\nu}(t),\theta,t\right)} \Psi\left(X^{\nu}(T),T\right) \right] \tag{19}$$

where, here, the expectation is taken w.r.t. path of $X^{\nu}$, which are the dynamics under the optimal policy under the invariant skill component of the cost. As a result of this, both can incrementally learned and using the previous results, the transfer of samples between varying tasks sharing a skill component.

## 5 Experimental Verification

### 5.1 Double Slit

To show Monte-Carlo approaches to artificial intelligence control, the double slit problem which is previously investigated is considered since it is, in one hand, is so simple to allow for a closed form solution for $\Psi$ to be obtained, but on the other hand, it is so complicated to underline the deficiencies of some previous approaches. It deals with a particle moving with constant velocity in one coordinate, and simultaneously, its position in an orthogonal direction is influenced by noise and controls. The goal of the task is that square error should be reduced to a target value at the end time whereas avoiding obstacles at some intermediate time. For this problem, the one dimensional dynamics are $dx = u + d\zeta$ and the cost is given by:

$$C_{\bullet}\left(x\right)=\omega\left(x-x_{target}\right)^{2}$$

$$C\left(x,t\right)=\begin{cases}10^{4} & if\ t=\dfrac{T}{2}\ \ and\ x\in Obstacle\\ 0 & else\end{cases}$$

where $\omega$ is a weight. In this regard, a discretisation with time step $0.02s$ is considered.

A comparison is made between the true optimal policy and those obtained using two variants of the proposed estimator, as $\bar{\Psi}_{OC}$, $\bar{\Psi}_{RL}$ which are based on single transitions from uniformly sampled start states and a $RL$ setting, learning from trajectory data without access to the cost, respectively. However, these used knowledge of the cost function to evaluate $\Phi$ in each step and the approach for sample sharing across time steps discussed, respectively. The low rank approximation and square exponential kernels $\psi\left(x,y\right)=\exp\left\{\left(x-y\right)^{2}/\gamma\right\}$ with $\gamma$, equal to the mean distance of the data are used for both cases. Moreover, two alternative approaches are considered for more comparison. The first alternative is the trajectory based Monte-Carlo approach. While it has the same number of trajectories as used in the RL setting, it uses a

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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Laplace approximation to the true $\Psi$ to attain a linear approximation of the optimal policy. The good performance of the proposed method can be clearly seen in which policies are considerably enhanced compared to those obtained by the alternative Monte-Carlo approach. However, the results of the proposed method are comparable to those obtained from the Laplace approximation. It should be pointed out that the results based on the Laplace approximation are computed by a prior knowledge about the true $\Psi$. Furthermore, it can be observed that the proposed approach makes better use of the sample provided by finding a policy which is applicable for varying starting positions. However, the multi-modality of the optimal policy is not recognized by the Monte-Carlo approach as improper results are obtained. For the variational approximation also the re-compute for each new starting location is very crucial.

The dependency of the estimate on the sample size is studied by comparing the evolution of the $L_1$ error of the estimates of $\Psi$ at time $t = 0$. Sample size is denoted as total number of transition samples seen. Therefore, the number of trajectories is the sample size divided by $100$ for $\bar{\Psi}_{RL}$. In order to also highlight the advantages of the sample re-use afforded by the approach in current paper, we also compare with $\bar{\Psi}$, the basic estimator given data of the same form as $\bar{\Psi}_{RL}$, i.e. recursive application of Eq. (16) without sample sharing across time steps.

### 5.2 Reaching Task of Arm Subspace

For simulating constrained tasks, reaching tasks on a subspace of the end-effector space of a torque controlled $5dof$ arm is considered. The considered skill component includes moving with the end-effector close to a two dimensional task space, while the task examples are given by specific reach targets. A linear subspace of the end effector space which is a non-linear subspace of the joint space is used in this section and the cost consists of two components:

$$C_{skill}(x,t) = \omega_{skill} \| J_{\varphi}(x) - j \|^2$$
$$C_{task}(x,\theta) = \omega_{task} \| \varphi(x) - \theta \|^2$$

where $\varphi(0)$ is the mapping from joint to end-effector coordinates, $J \,\&\, j$ define the task subspace, $\theta$ specifies the reaching target and $\omega's$ are weights. A position control over a $2s$ horizon with a $0.02\text{s}$ discretisation is considered here, again.

Due to the restrictions of low cost trajectories to a small subspace, this task is challenging for sample based approaches. Although some researchers were suggested to improve end-effector exploration an inverse dynamics policy can be used, it is necessary to increase sample sizes for overcoming this problem. With focus on the case of changing targets, the ideas originated are used as it is assumed that the operators have been estimated under the skill augmented dynamics (as an alternative to explicit sample generating, utilizing the importance sample based estimator and then, collecting a sample under $X^0$, which is more time consuming than what that is performed here), and then, the subsequent learning for a novel task is considered by the estimator, utilizing the already estimated operators in two ways. They are directly used in the calculation of $\bar{\Psi}$; however, it should be noted that the trajectories can be sampled without considering a specific policy since these are only required to provide $D'$. As a result, the policy arising is used only when considering $C_{skill}$, i.e., the skill policy associated with $\bar{\Psi}$ computed using the given operators and $C_{task}(0) = 0$.

It is demonstrated that sampling under the skill policy is more effective in exploring the task relevant sub space than null policy.

## 6 Summary and Conclusion

An innovative approach is presented in the current paper to solve stochastic optimal control problems which have the artificial intelligence control form

International Journal of Scientific & Engineering Research, Volume 6, Issue 1, January-2015
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using Monte-Carlo estimates of operators arising from a RKHS embedding of the problem. It leads to a consistent estimate of problem $\Psi$. Although direct application of Monte-Carlo estimation to point evaluation of $\Psi$ also obtains a consistent estimate, a trajectory sample for each state at which an action is to be computed is needed due to impracticality of computing the controls for anything but simple problems. Despite some suggestions provided by previous works to reduce sample complexity, it is demonstrated that the proposed approach is of more generality in policy than these previous suggestions which are not consistent in their processes. Moreover, it is shown that sample re-using is possible by the proposed estimators, particularly for cases in which a new sample set is needed in advance. Particular emphasis is focused on the transfer in cases where execution of several, potentially related, tasks on the same plant is required. It shows that use of samples from all tasks to learn invariant aspects are possible. An alternative approach of the proposed method can be defined to combine solutions to local control problem, as defined by $\Phi$, and hence, to solve a more complicated large scale problem. Combining the proposed methods with alternative variation approaches will be followed in future studies so that a good estimate for the comparatively simpler local problems can be achieved. Although kernel is not chosen in this study, making informed kernel choices based on in advance knowledge about the structure of the problem may be positive. In current work, a method of utilizing RKHS embeddings of the transition probability in computing the value functions in MDPs. However, this work is of some advantages over their work. The first advantage is that the optimal controls are directly obtained in this paper instead of computing the value function. It is better than use of explicit maximization to obtain optimal controls. In addition, in current study focused on finite state problems (where computation of the optimal $u$ is simpler), while harder continuous problem are studied in this paper which provides convergence guarantees in this setting. As the final point, it should be noted that the structure of the problem is used to efficiently estimates the required quantities which leads to efficient sample re-use and transfer.

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